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An Axiomatic Approach to Voronoi-Diagrams in 3D

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Voronoi-diagrams were first introduced for sets of points, and later generalized to other sets. There have been very few attempts to generalize Voronoi-diagrams for non-point sets to three (or higher) dimensions. We define a Voronoi-diagram for a quite general subset of three-space which is characterized by specific axioms. This diagram is one-dimensional and connected provided that the complement of the subset is connected. These Voronoi-diagrams can be used in retraction methods to solve the Findpath Problem for a ball moving in a three-dimensional environment. © 1991 Academic Press, Inc.

INTRODUCTION

Voronoi-diagrams were first introduced in [13] for finite sets $\mathscr{A}$ of points in the plane. They were then generalized by many authors, e.g., Refs. [1, 3, 6] to finite sets of points in more than two dimensions, and to non-point sets $\mathscr{A}$, e.g., Refs. [7, 11, 14]. The Voronoi-diagram of $\mathscr{A}$ can equivalently be referred to as that of the complement, $\mathscr{F}$, of $\mathscr{A}$, where the complement is possibly taken w.r.t. a subset of the space considered. (We also use it in that sense below.) However, there exist very few approaches for non-points sets $\mathscr{A}$ in three (or higher) dimensions, e.g., Refs. [4, 5, 9].

Generalizations to three and higher dimensions are of enormous importance for advanced applications in geometric engineering. For example, in robotics, the Voronoi-diagram for a set $\mathscr{F}$ can be used to reduce the problem of finding a collision-free path for a point through $\mathscr{F}$ to searching for a path on the graph specified by the Voronoi-diagram. This idea was exploited first in [10]. For this purose the following theorem must be shown:

For each p, q on the Voronoi-diagram of $\mathscr{F}$: If there exists a connected curve from p to q in $\mathscr{F}$ then there exists one on the Voronoi-diagram of $\mathscr{F}$; for all $p \in \mathscr{F}$ there exists a connected curve in $\mathscr{F}$ from p to the Voronoi-diagram of $\mathscr{F}$.

The approach can be extended for moving a circle or a sphere in $\mathscr{F}$. In some

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approaches, e.g. [9], the configuration space of the moving object is taken as set $\mathscr{F}$; this makes it possible to consider also objects that have an orientation.

The diagram to be defined is a generalization of the medial axis of a contour in the plane, considered in [2, 8]. Our approach to Voronoi-diagrams is different from and more general than all known approaches. We are working in three-dimensional Euclidean space and can handle sets $\mathscr{F}$ whose boundaries are surfaces and, in fact, quite general surfaces. In order to obtain maximal generality we give an axiomatic characterization of the respective contour of $\mathscr{F}$. A precise description of the contour suffices to fix the diagram. Changes in the description of the contour of $\mathscr{F}$ might change the diagram. Roughly, there must exist a finite set contour($\mathscr{F}$) with the following properties:

(1) contour($\mathscr{F}$) is a partition of the boundary of $\mathscr{F}$.

(2) Moving on a line normal to some element in contour($\mathscr{F}$) away from that element the distance to the element increases strictly.

(3) Points with equal distances to two elements of contour($\mathscr{F}$) assume the maximal distance to the boundaries of the elements.

(4) Each point in $\mathscr{F}$ is uniquely projectable onto each element in contour($\mathscr{F}$).

(5) All elements in contour($\mathscr{F}$) are continuously differentiable.

If $\mathscr{F}$ is a polyhedral set of $\mathbf{R}^3$ then, usually, the set of faces without edges, together with the edges without endpoints, together with the endpoints is such a set contour($\mathscr{F}$). There may be some further splitting necessary in order to guarantee all the properties required.

The generalization of the Voronoi-diagram is done in two steps: (1) A two-dimensional set is introduced, the "2-skeleton." (2) The Voronoi-diagram (or 1-skeleton), the intersection of the surfaces in the 2-skeleton, is introduced. It seems to be quite natural to do the generalization in these two steps. The "retraction" of points in $\mathscr{F}$ onto the Voronoi-diagram is also done in two steps, i.e., by first retracting the point onto the 2-skeleton and, in a second step, onto the Voronoi-diagram.

Roughly, the Voronoi-diagram of such a set $\mathscr{F}$ is the set of points that have an equal distance to three elements in the contour of $\mathscr{F}$; this entails that the Voronoi-diagram is a one-dimensional manifold in $\mathscr{F}$. (One has to be careful in defining the set of points that have an equal distance to three elements in the contour for the cases in which one of the elements is contained in the closure of a second one, in order to keep the Voronoi-diagram one-dimensional.) We show that the Voronoi-diagrams for such sets $\mathscr{F}$ that are bounded are applicable to the Findpath Problem; i.e., we prove the above theorem. For this purpose a function is defined that projects each point in $\mathscr{F}$ onto the Voronoi-diagram. In contrast to generalizations in the plane this mapping need not be continuous in points that are already on the Voronoi-diagram. However, for such points we are able to show that there is a path on the Voronoi-diagram between the point and each lines of projections of sequences converging to that point. Together with the continuity in all other points this yields the desired theorem.

NOTATIONS

In order to generalize the Voronoi-diagram to three dimensions a lot of definitions and notations are needed. These definitions and notations are, although intuitively easy to understand, quite technical.

Notation. A is a *curve* (a *face*) iff A is the image of a continuously differentiable function from $(0, 1)$ $((0, 1)\times(0, 1))$ to $\mathbf{R}^3$, and A is not a point (not a curve or a point).

The distance of two sets A, B, $\operatorname{dist}(A, B)$, is the infimum of (Euclidean) distances of points p, q, with $p\in A$ and $q\in B$.

For two points p, q, we denote the straight line containing p and q by pq, the straight line segment connecting p and q by $\overline{pq}$.

The *surface containing A normal to B* is the set $\{p\,|\,\exists q\in A\colon pq \text{ normal to } B\}$.

A point p is a *singular point* of a set S iff $p\in S$, and there exists an open neighborhood U of p with $U\cap S=\{p\}$.

By $\mathring{A}$, $\bar{A}$, ∂A, the interior, the closure, and the boundary of A are denoted.

Let $\mathscr{F}$ first be any subset of $\mathbf{R}^3$. The following definitions are needed to specify which of the elements are in the contour of $\mathscr{F}$. In order to apply the Voronoi-diagram to the Findpath Problem, especially in order to prove the Main Theorem on Voronoi-diagrams, the elements in the contour of $\mathscr{F}$ have to satisfy certain conditions. In our approach, we try to single out a set of properties of $\mathscr{F}$ that are indispensable for the proof of the Main Theorem. These conditions are satisfied, for example, if $\mathscr{F}$ is bounded by planar faces and straight line segments; see [12]. It should be easy to prove that sets $\mathscr{F}$ bounded by other simple faces, e.g. (parts of) spheres, ellipsoids, or superellipsoids, satisfy these conditions.

DEFINITION.

A is removing $:\Leftrightarrow$ (A is a point) or
$(\forall p, l)$: (if $p\in A$ and l is a line through p normal to A then the distance to A increases when moving away from p along l).

p is *uniquely projectable onto A*
$:\Leftrightarrow$ there exists at most one $q\in\bar{A}$ with $\operatorname{dist}(q, p)=\operatorname{dist}(A, p)$, pq normal to A.

$$F(A, B) := \begin{cases} \text{the surface containing } A \text{ normal to } B & \text{if } A\subseteq\bar{B},\\ \text{the surface containing } B \text{ normal to } A & \text{if } B\subseteq\bar{A},\\ \{p\,|\,\exists q_1\in A, q_2\in B\colon \operatorname{dist}(A, p)=\operatorname{dist}(q_1, p)=\operatorname{dist}(q_2, p)=\operatorname{dist}(B, p),\ q_1p \text{ normal to } A,\ q_2p \text{ normal to } B\} & \text{otherwise.}\end{cases}$$

face(A, B) := the closure of $(F(A, B) \cap \bar{\bar{\mathscr{F}}})$-
$\{p \mid p$ is a singular point of $F(A, B)\}$.

vface(A, B) := $\{p \mid p \in \text{face}(A, B), \text{dist}(A, p) \leqslant \text{dist}(\partial\mathscr{F}, p)\}$.

curve (A, B, C) := face $(A, B) \cap \text{face}(A, C) \cap \text{face}(B, C)$.

vcurve(A, B, C) := $\{p \mid p \in \text{curve}(A, B, C)$,
$\forall E \in \text{contour}(\mathscr{F})): \text{dist}(E, p) \geqslant \text{dist}(A, p)\}$.

vertex(contour$(\mathscr{F})$) := $\{p \mid A, B, C, A', B', C' \in \text{contour}(\mathscr{F})$,
p an endpoint of a connected component of
vcurve$(A, B, C) \cap$ vcurve$(A', B', C')\}$.

s, t are *adjacent vertices of* vcurve(A, B, C)
:⇔ $s \neq t$, s, $t \in$ vcurve(A, B, C),
s, $t \in$ vertex(contour$(\mathscr{F})$),
vcurve(A, B, C) is connected between s and t, and
$(\forall p \in \text{vcurve}(A, B, C)$, p between s and $t)$:
$p \notin$ vertex(contour$(\mathscr{F})$).

near(A, B) :⇔ $B \subseteq \bar{A}$ or $A \subseteq \bar{B}$.

Let C be a connected component of vface(A, B).

$s_{A,B;C}$:= such that $s_{A,B;C} \in C$, and
$\text{dist}(A, s_{A,B;C}) = \min\{d \mid d = \text{dist}(A, p)$, for some $p \in C\}$.

(The minimum exists because each connected component of the face is closed.)

$$N_{a,A,B} := \begin{cases} \text{the closure of } \{p \mid \exists q \in as_{A,B;C}\colon pq \text{ is normal to } A, \\ q \text{ uniquely projectable onto } A\} \\ \quad \text{if } a \neq s_{A,B;C}, \\ \text{the closure of } \{p \mid \exists q \in a's_{A,B;C}\colon pq \text{ is normal to } A, \\ q \text{ uniquely projectable onto } A\}, \\ \quad \text{for some } a' \in C,\ a's_{A,B;C} \text{ not normal to } A, \\ \quad \text{if } a = s_{A,B;C}, \\ \text{where } C \text{ is the connected component of vface}(A, B) \\ \text{with } a \in C. \end{cases}$$

A *and* B *have no maximal distance w.r.t.* $\mathscr{F}$
:⇔ $A = B$ or both, A and B, are points, or near (A, B), or
[(A is a curve and B is a point or a curve,
or A is a face) and
(for each connected component C of vface(A, B),
and each $a \in C$:
$f_a\colon C \cap N_{a,A,B} \to \mathbf{R}$, $p \to \text{dist}(A, p)$
has no strict relative maximum)] or
[(B is a curve and A is a point or a curve,
or B is a face) and
(for each connected component C of vface(B, A),
and each $a \in C$:
$f_a\colon C \cap N_{a,B,A} \to \mathbf{R}$, $p \to \text{dist}(B, p)$
has no strict relative maximum)].

A and B *generate a convex face w.r.t.* $\mathscr{F}$
:⇔ near(A, B), or
($\forall p \in E$: $|N_{p,A,B} \cap F| > 2 \Rightarrow N_{p,A,B} \cap E \subseteq F$,
where E is a connected component of
$\{p \mid p \in \text{face}(A, B), \text{dist}(A, p) \leqslant \text{dist}(\partial\mathscr{F}, p)\}$,
$F := \{p \in E \mid p \in \text{face}(A, C) \cap \text{face}(B, C)$,
for some $p \in \text{contour}(\mathscr{F})\}$).

We now draw restrictions on $\mathscr{F}$ by requiring that its boundary admits a partition contour($\mathscr{F}$) as follows:

DEFINITION. contour ($\mathscr{F}$) is a set S of faces, curves, and points such that

$\forall A \in S$: ($A \subseteq \partial\mathscr{F}$, A connected, A is removing,
($\forall p \in \mathscr{F}$): p is uniquely projectable onto A
—except for finitely many p—,

$\forall A, B \in S, A \neq B$: ($A \cap B = \varnothing, \bar{A} \cap B \neq \varnothing \Rightarrow B \subset \bar{A}$,
A and B have no maximal distance,
A and B generate a convex face w.r.t. $\mathscr{F}$,

$$\bigcup_{A \in S} A = \partial\mathscr{F}.$$

The condition that all A and B generate a convex face w.r.t. $\mathscr{F}$ can be weakened, because it is not needed for all A, B. For details see [12].

Remarks. (1) contour($\mathscr{F}$) is not unique because of the possibility of splitting. Splitting some A in contour($\mathscr{F}$) further does not influence the correctness of the theorems stated below, but will increase the number of vertices and curves in the Voronoi-diagram. These additional vertices and curves are not necessary for the proof of the Main Theorem; they only increase the time to compute the Voronoi-diagram.

(2) Intuitively, contour($\mathscr{F}$) is the set of

- faces in $\partial\mathscr{F}$ without bounding curves, together with
- the bounding curves of the faces and the other curves in $\partial\mathscr{F}$ without endpoints, together with
- the endpoints of the curves and the other points in $\partial\mathscr{F}$.

(3) The requirement that all elements in contour($\mathscr{F}$) are removing is needed for the retraction onto the diagram. For example, in the two-dimensional approach circles must be split into quarters of circles. In three dimensions one may, e.g., have at most the eighth parts of spheres in contour($\mathscr{F}$).

(4) The requirement that almost all p are uniquely projectable onto A (for all $A \in \text{contour}(\mathscr{F})$) is necessary to keep the retraction continuous. This condition cannot always be reached by splitting a face or curve.

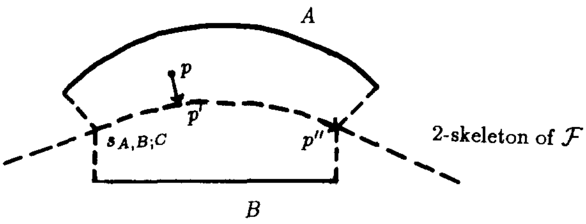

FIG. 1. A and B have a maximal distance.

(5) There must not be a point with maximal but equal distance to some pair of elements $A, B \in \text{contour}(\mathscr{F})$ because this would cause troubles when retracting a point on the 2-skeleton onto the Voronoi-diagram. More concretely, the distance to contour($\mathscr{F}$) would not necessarily increase by that process. In Fig. 1, A and B have a maximal distance. The result of the retraction of p onto the 2-skeleton of $\mathscr{F}$ is p', the result of the retraction of p' onto the Voronoi-diagram of $\mathscr{F}$ is p''. However, $\text{dist}(A, p') \geqslant \text{dist}(A, p'')$. Thus, a sphere $\mathscr{M}$ centered at p' may be totally in $\mathscr{F}$, while $\mathscr{M}$ centered at p'' may not totally be in $\mathscr{F}$.

(6) $s_{A,B;C}$, and $N_{p,A,B}$ are necessary to specify the retraction of a point p on the 2-skeleton onto the Voronoi-diagram. This retraction is specified as the movement of p along the intersection of $N_{p,A,B}$ and the 2-skeleton of $\mathscr{F}$. For the purpose of proving that the retraction is continuous, an ordering on the elements in contour($\mathscr{F}$) is used. This ordering makes it easy to uniquely specify A and $B \in \text{contour}(\mathscr{F})$ such that $p \in \text{vface}(A, B)$ and p can be moved along the intersection of $N_{p,A,B}$ and the 2-skeleton of $\mathscr{F}$ in such a way that the distance to $\partial\mathscr{F}$ does not decrease.

In the following we presuppose that $\mathscr{F}$ is an open connected subset of three-dimensional Euclidean space, $\mathscr{F} \neq \varnothing$, such that contour($\mathscr{F}$) is finite.

VORONOI-DIAGRAMS

Before defining the 2-skeleton and the Voronoi-diagram of a set $\mathscr{F}$ we give some properties of faces and curves that are needed for the Main Theorem.

LEMMA ON FACES AND CURVES. (a) $\forall A, B \in \text{contour}(\mathscr{F})$, $A \neq B$: face(A, B) *is a subset of the finite union of images of continuous functions on* $(0, 1) \times (0, 1)$.

(b) $\forall A, B, C \in \text{contour}(\mathscr{F})$, *pairwise distinct*: curve(A, B, C) *is a subset of the finite union of continuous functions on* $(0, 1)$.

(c) *Bounding curves of connected components of* vfaces *are contained in* vcurves.

(d) *Endpoints of connected components of* vcurves *are vertices.*

The proof of the lemma is given in the appendix.

We are now in the position to define the 2-skeleton and the Voronoi-diagram for a set $\mathscr{F}$. (The 2-skeleton and the Voronoi-diagram actually depend on contour($\mathscr{F}$). However, we denote them by 2-skel($\mathscr{F}$) and vor($\mathscr{F}$) to make things more readable.)

DEFINITION.

2-skel($\mathscr{F}$) (the *2-skeleton* of $\mathscr{F}$)

$$:= \{p \mid p \in \text{vface}(A, B), \text{ for some } A, B \in \text{contour}(\mathscr{F}), A, B \text{ pairwise distinct}\}.$$

vor($\mathscr{F}$) (the *Voronoi-diagram* of $\mathscr{F}$)

$$:= \{p \mid p \in \text{vcurve}(A, B, C), \text{ for some } A, B, C \in \text{contour}(\mathscr{F}), A, B, C \text{ pairwise distinct}\}.$$

As an immediate consequence of this definition and the definitions of vcurve and vface one gets vor($\mathscr{F}$) = 1-skeleton of $\mathscr{F} \subseteq$ 2-skel($\mathscr{F}$) $\subseteq \bar{\mathscr{F}}$.

See Fig. 2 for an example of a Voronoi-diagram. There, $\mathscr{F}$ is the interior of a cube where a smaller cube in the front is left out. The faces spanned by the edges

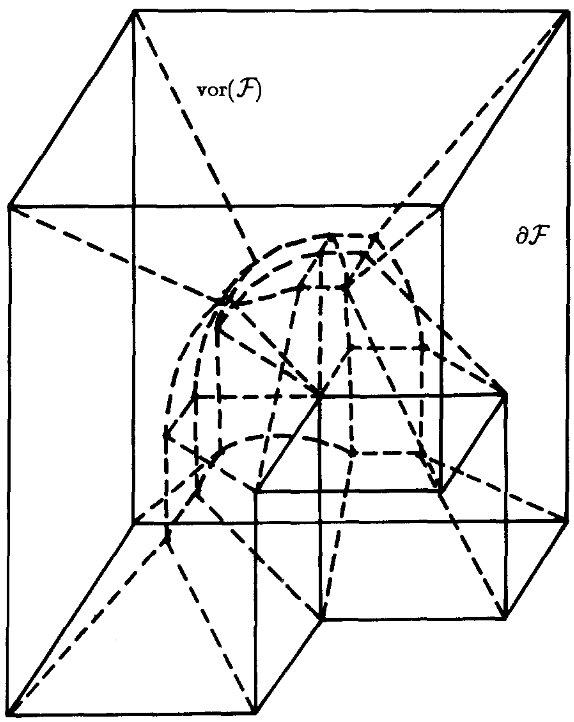

FIG. 2. Example of a Voronoi-diagram.

of the Voronoi-diagram and the edges of the polygons in the boundary of $\mathscr{F}$ build the 2-skeleton of $\mathscr{F}$.

Presupposition. Let vertex(contour($\mathscr{F}$)) be finite.

The fact that vertex(contour($\mathscr{F}$)) is finite is crucial for the application of a graph search technique to the Voronoi-diagram, which has vertex(contour($\mathscr{F}$)) as the set of vertices. (It seems to be very likely that the finiteness of vertex(contour($\mathscr{F}$)) already is a consequence of the other requirements on $\mathscr{F}$.)

So far, we have specified the notion of a Voronoi-diagram for a three-dimensional, connected, open subset $\mathscr{F}$ for which a set contour($\mathscr{F}$) exists that satisfies the requirements presupposed. The rest of this subsection prepares the proof of the Main Theorem on Voronoi-diagrams, the theorem that guarantees the existence of a collison-free path for a sphere $\mathscr{M}$ on the Voronoi-diagram whenever a collision-free path for $\mathscr{M}$ in $\mathscr{F}$ exists. The proof of the Main Theorem is based on a retraction of each collision-free path for $\mathscr{M}$ in $\mathscr{F}$ onto the Voronoi-diagram. The next paragraphs describe this retraction and some properties of it that are needed in the proof of the Main Theorem.

Next we give the definitions of two functions. The first, retr-1, maps the closure of $\mathscr{F}$ onto the 2-skeleton of $\mathscr{F}$. The second one, retr-2, maps the 2-skeleton of $\mathscr{F}$ onto the Voronoi-diagram of $\mathscr{F}$. Both functions are almost retractions. (That is, both are almost continuous and are the identity on 2-skel($\mathscr{F}$) and on vor($\mathscr{F}$), respectively.) The consecutive application of these two functions will be used to map each point in $\mathscr{F}$ onto the Voronoi-diagram of $\mathscr{F}$.

The definitions of retr-1 and retr-2 need two more definitions: V1 and V2 Roughly, by V1 and V2 the direction of the retraction is specified. The retraction of a point p onto the 2-skeleton can always be done by moving on a straight line normal to the element in the contour of $\mathscr{F}$ to which the point p is closest. The retraction of a point on the 2-skeleton onto the Voronoi-diagram cannot be done that easily. This is because the point has to stay on the 2-skeleton during the retraction in order to guarantee that it will not come too close to some other element in the contour of $\mathscr{F}$. Note that $\mathrm{V1}(p)$ is a point, while $\mathrm{V2}(p, A, B)$ is a curve.

DEFINITION.

$$\begin{aligned} \mathrm{V1}: \quad & \bar{\mathscr{F}} \to \partial\mathscr{F}, \\ \mathrm{V1}(p) := \ & \text{a } q \in \partial\mathscr{F} \text{ such that } \operatorname{dist}(p, q) = \min\{\operatorname{dist}(p, q) \mid q \in \partial\mathscr{F}\}, \\ & pq \text{ normal to } A, \text{ where } q \in A \in \operatorname{contour}(\mathscr{F}),\ \operatorname{dist}(p, q) = \operatorname{dist}(p, A). \end{aligned}$$

(We use the notion "$\mathrm{V1}(p)$ is uniquely specified" in case there is only one q satisfying the condition in the definition of V1.)

$\mathrm{V2}(p, A, B)$ is defined in the following table for $p \in \mathrm{vface}(A, B)$. (In the table a "+" in column near indicates near(A, B), a "−" indicates $\neg$near(A, B). Let C be the connected component of vface(A, B) with $p \in C$.)

A	B	near	$V2(p, A, B)$
Point	Point	–	(Straight line containing p and $s_{A,B;C}) \subseteq F(A, B)$
Curve	Point	+	Straight line containing p normal to A
Face	Point	+	Straight line containing p normal to A
Face	Curve	+	Straight line containing p normal to A
Curve	Point	–	$N_{p,A,B} \cap \text{face}(A, B)$
Curve	Curve	–	$N_{p,A,B} \cap \text{face}(A, B)$
Face	Point	–	$N_{p,A,B} \cap \text{face}(A, B)$
Face	Curve	–	$N_{p,A,B} \cap \text{face}(A, B)$
Face	Face	–	$N_{p,A,B} \cap \text{face}(A, B)$

Let $\prec$ be some linear ordering on contour($\mathscr{F}$) such that: "faces $\prec$ curves $\prec$ points." (We use the notation A, B *are less than* A', B' *w.r.t.* $\prec$ iff $(A \prec A')$ or $(A = A'$ and $B \prec B')$.)

DEFINITION. Let $\mathscr{F}$ be bounded.

$$\text{retr-1}: \quad \mathscr{F} \to \text{2-skel}(\mathscr{F}),$$

where retr-1(p) is the first intersection point with 2-skel($\mathscr{F}$) when moving on the straight line through p and V1(p) from V1(p) to p, if $q \notin$ 2-skel($\mathscr{F}$). retr-1$(p) = p$, otherwise.

$$\text{retr-2}: \quad \text{2-skel}(\mathscr{F}) \to \text{vor}(\mathscr{F}).$$

Let $p \in \text{vface}(A, B)$ with $A \prec B$, A, B the least such elements in contour($\mathscr{F}$) w.r.t. $\prec$. Let C be the connected component of vface(A, B) with $p \in C$. retr-2(p) is the first intersection point with vor($\mathscr{F}$) behind p when moving on V2(p, A, B) from $s_{A,B;C}$ to p if $p \notin$ vor($\mathscr{F}$). retr-2$(p) = p$, otherwise.

$$\text{retr}: \quad \mathscr{F} \to \text{vor}(\mathscr{F})$$

where retr(p) = retr-2(retr-1(p)).

Figure 3 illustrates these definitions by an example. There, only part of $\mathscr{F}$ is shown to make the figure more clear. The $\mathscr{F}$ considered in Fig. 3 is the same as in Fig. 2, so one can look up the whole $\mathscr{F}$ and its Voronoi-diagram if necessary.

The three functions, retr-1, retr-2, and retr need not map onto 2-skel($\mathscr{F}$) and vor($\mathscr{F}$), respectively, if $\mathscr{F}$ is not bounded; i.e., the intersection point need not exist for unbounded $\mathscr{F}$. However, in case $\mathscr{F}$ is bounded there must be the required intersection point with 2-skel($\mathscr{F}$) and vor($\mathscr{F}$), respectively. Furthermore, the condition that $p \in \text{vface}(A, B)$ and $A \prec B$, A, B the least such elements in contour($\mathscr{F}$) w.r.t. $\prec$, specifies A and B uniquely for each $p \in \mathscr{F} - \text{vor}(\mathscr{F})$.

V1 is defined to choose an appropriate q for each p. V1(p) is specified uniquely whenever $p \notin$ 2-skel($\mathscr{F}$) except at finitely many points. V2 is specified uniquely for

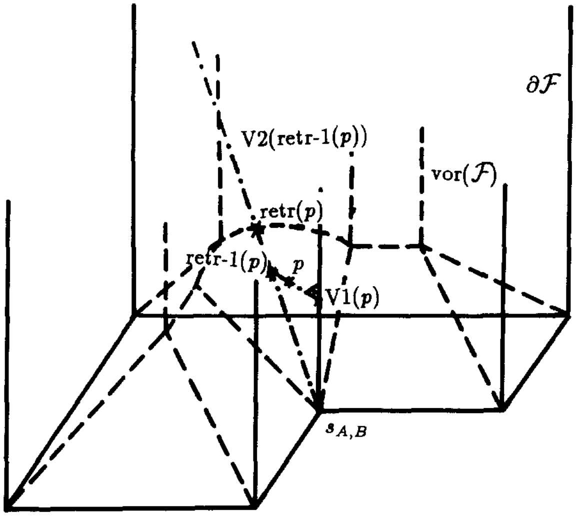

FIG. 3. Example for the retraction map.

each $p \notin \text{vor}(\mathscr{F})$ once the point $s_{A,B;C}$ is fixed and $p \neq s_{A,B;C}$. From this one gets that retr-1 and retr-2, and, hence, retr, are specified uniquely except at finitely many points, and points p with $p \in \text{2-skel}(\mathscr{F})$, $\text{retr-1}(p) \in \text{vor}(\mathscr{F})$, or $\text{retr-1}(p) = s_{A,B;C}$ for some $A, B \in \text{contour}(\mathscr{F})$, $A \neq B$.

Using the maps retr-1 and retr-2, one can show that 2-skeletons and Voronoi-diagrams are closed. This fact is needed to prove that the map retr is almost continuous. It is clear from the definition of retr-1 and retr-2 and the considerations above that retr-1 and retr-2, hence, retr, are the identity on vor($\mathscr{F}$).

LEMMA ON CONTINUITY OF retr. *Let $\mathscr{F}$ be bounded. Let p be such that* $p \notin \text{2-skel}(\mathscr{F})$, $\text{retr-1}(p) \notin \text{vor}(\mathscr{F})$, $\text{retr-1}(p) \neq s_{A,B;C}$, *for all* $A, B \in \text{contour}(\mathscr{F})$, $A \neq B$, *C a connected component of* vface(A, B). *Then* retr *is continuous in p, except for finitely many points.*

The proof of the lemma is given in the appendix.

The Lemma on Adjacent Vertices shows that between two vertices on a curve there is another vertex, or all points belong to the Voronoi-diagram, or no point belongs to the Voronoi-diagram.

LEMMA ON ADJACENT VERTICES. *Let $\mathscr{F}$ be bounded,* $p \in \text{vcurve}(A, B, C)$, $p \notin \text{vertex}(\text{contour}(\mathscr{F}))$, $p \notin \partial\mathscr{F}$. *Then*

$(\exists s, t)$: $s, t \in \text{vcurve}(A, B, C)$,

s, t are adjacent vertices of vcurve(A, B, C), *and*

p is between s and t.

Proof. This lemma is an immediate consequence of the Lemma on Faces and Curves, part (d). ▌

Four vcurves meet in each vertex of the diagram. In case a connected component of a vcurve is only a point three more vcurves will meet in the corresponding vertices.

As already stated, the map retr may not be continuous in points that are in 2-skel($\mathscr{F}$) and points that are mapped onto vor($\mathscr{F}$) or some $s_{A,B;C}$ by retr-1, and at finitely many other points. retr can be made continuous in these finitely many other points. In the following we assume that this has already been done for retr. Hence, in the retraction of a path there remain only some gaps that may end on the Voronoi-diagram. We show below that in such situations one will put part of the vcurve that contains p to the retraction of the path. Points p with retr-1$(p)=s_{A,B;C}$ will be avoided by taking a path in a sufficiently small neighborhood of the actual path.

The next lema shows that the distance does not decrease when retracting a point in $\mathscr{F}$ onto the Voronoi-diagram by the map retr. This entails that the retraction of a collision-free path is collision-free. Together with the continuity of retractions and the closedness of 2-skeletons and Voronoi-diagrams this gives the Main Theorem on Voronoi-Diagrams.

LEMMA ON INCREASING DISTANCES. *Let* $p\in\mathscr{F}$. *Then* $\mathrm{dist}(\partial\mathscr{F}, p)\leqslant\mathrm{dist}(\partial\mathscr{F}, \mathrm{retr}(p))$.

The proof of the lemma is given in the appendix.

To state the Main Theorem two more notations are needed:

Let $p\in\mathscr{F}$. Let $\mathscr{M}_r$ denote the sphere with radius r. Then $\mathscr{M}_r(p)$ is the sphere $\mathscr{M}_r$ positioned at p.

Let S, T be paths with $S\cap T=\{p\}$ for some p. Then $S\circ T$ is the connection of these two paths at p.

We are now in the position to formulate and prove the Main Theorem, the theorem that guarantees the existence of a path on the diagram whenever there exists a path through $\mathscr{F}$. By this theorem it is especially guaranteed that the Voronoi-diagram of a connected set $\mathscr{F}$ (satisfying the properties on contour($\mathscr{F}$) required) is connected.

MAIN THEOREM ON VORONOI-DIAGRAMS. *Let* $p_1, p_2\in\mathscr{F}$ *be such that* $\mathscr{M}(p_1)$, $\mathscr{M}(p_2)\subseteq\mathscr{F}$. *There is a collision-free path S from* p_1 *to* p_2 *for* $\mathscr{M}$ *through* $\mathscr{F}$ *iff there is a collision-free path V from* $\mathrm{retr}(p_1)$ *to* $\mathrm{retr}(p_2)$ *for* $\mathscr{M}$ *such that* $V\subseteq\mathrm{vor}(\mathscr{F})$.

Figure 4 shows an example of the application of the Voronoi-diagram to the Findpath Problem. There $\mathscr{F}$ is the interior of a cube with a separating plane from the bottom to half of the height. In this example the retraction of the path S of $\mathscr{M}$

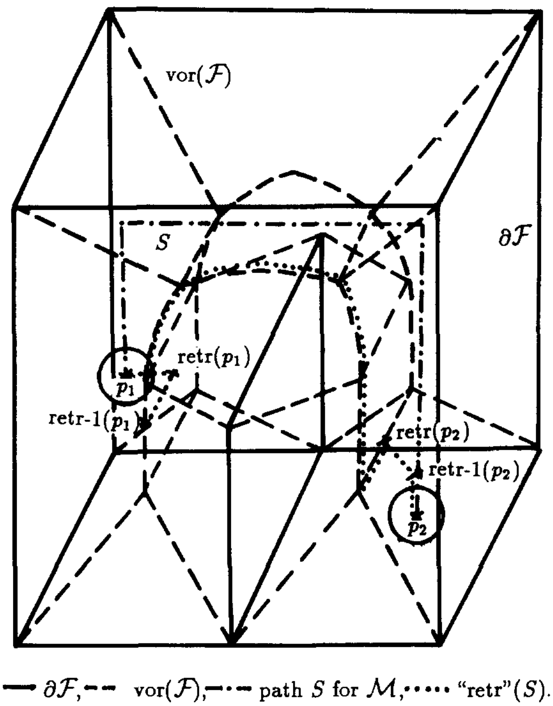

FIG. 4. Application of the Voronoi-diagram to the Findpath Problem.

through $\mathscr{F}$ is already made connected, as is shown in the proof of the Main Theorem.

Proof. Let V be a collision-free path from retr(p_1) to retr(p_2) for $\mathscr{M}$ with $V \subseteq \text{vor}(\mathscr{F})$. Then $S = (\overline{p_1 \text{ retr-1}(p_1)} \circ \text{V2}(\text{retr-1}(p_1))$ between p_1 and $\text{retr}(p_1) \circ V \circ \text{V2}(\text{retr-1}(p_2))$ between p_2 and $\text{retr}(p_2) \circ \overline{p_2 \text{ retr-1}(p_2)})$ is a collision-free path from p_1 to p_2 for $\mathscr{M}$ through $\mathscr{F}$: By the proof of the Lemma on Increasing Distances the distance to $\partial\mathscr{F}$ increases when moving on a straight line from p to retr-1(p) for all $p \in \mathscr{F}$ and it increases when moving on V2(q) from q to retr-2(q) for all $q \in$ 2-skel($\mathscr{F}$). Hence, S is collision-free. S is also connected because all five parts of it are connected: V is connected by assumption; the other four parts of S are connected as an immediate consequence of their definitions.

On the other hand, let S be a collision-free path from p_1 to p_2 for $\mathscr{M}$ through $\mathscr{F}$. Let U be the set of points in S for which retr-1 is not continuous. Let S' be a collision-free path for $\mathscr{M}$ in a sufficiently small neighbourhood of S which avoids the points in $U -$ 2-skel($\mathscr{F}$). Such a path always exists because $\mathscr{F}$ is open and $U -$ 2-skel($\mathscr{F}$) is finite. retr-1(S') is continuous except at points in vor($\mathscr{F}$). However, for each $p \in \text{retr-1}(S') \cap \text{2-skel}(\mathscr{F})$ and each sequence (p_n) in $\mathscr{F} -$ 2-skel($\mathscr{F}$) converging to p the straight line segment $\overline{p(\lim \text{retr-1}(p_n))}$ is contained in 2-skel($\mathscr{F}$); see the proof of the Lemma on Continuity of retr. Let S_1 be retr-1(S') together with these straight line segments. (Note that lim retr-1(p_n) is independent of the sequence (p_n) considered; compare the proof of the Lemma on Continuity of retr.) Let $U' = \{s_{A,B;C} \mid A, B \in \text{contour}(\mathscr{F}), C$ a connected component of vface$(A, B)\}$. Let S'' be a collision-free path for $\mathscr{M}$ in a sufficiently small neighbourhood of S_1 which avoids the points in U'. Such a path always exists because $\mathscr{F}$ is open and U' is finite. Let $V' = \text{retr-2}(S'')$. V' is collision-free by the Lemma on Increasing Distances (and continuity of dist). However, V' need not be connected, but it consists of connected components who's endpoints are on vor($\mathscr{F}$). Let $U'' = S'' \cap \text{vor}(\mathscr{F})$. W.l.o.g. assume that S'' intersects vor($\mathscr{F}$) only in finitely many points. Let $p \in U''$. Let (p_n) be some sequence in some connected component of $S'' \cap \text{vface}(A, B)$ converging to p such that A, B are the two elements in contour($\mathscr{F}$) that are used for retr-2(p_n). (Note that this is always possible because (p_n) is a sequence that is totally contained in one connected component of a vface. Furthermore, lim(retr-2(p_n)) does not depend on the sequence as long as it is in one connected component of vface(A, B); see the proof of the Lemma on Continuity of retr.) As in the proof of the Lemma on Continuity of retr one gets that there is a point q on $\text{vor}(\mathscr{F}) \cap \text{V2}(p, A, B)$ on the same side of p as lim retr-2(p_n). Because A and B generate a convex face w.r.t. $\mathscr{F}$, one gets $\text{V2}(p, A, B) \cap \text{vface}(A, B) \subseteq \text{vor}(\mathscr{F})$. ($p$, lim retr-2($p_n$), $q \in \text{V2}(p, A, B) \cap \text{vface}(A, B)$.) (Note that $s_{A,B;C} \neq p$, retr-2$(p_n) \neq p$ because retr-2 is not continuous in p.) So, moving on V2 is the same as moving on vor($\mathscr{F}$) in this region. Connect V' by V2(p, A, B) between p and lim(retr-2(p_n)). Do the same also for the other connected component of 2-skel($\mathscr{F}$) which contains p. Doing so for all $p \in U''$ one gets a collision-free path, V, $V \subseteq \text{vor}(\mathscr{F})$, from retr($p_1$) to retr($p_2$). ▮

This Main Theorem already shows that the algorithm for constructing a collision-free path from p_1 to p_2 for $\mathscr{M}$ through $\mathscr{F}$ can be carried over from two-dimensional space (see, e.g., [14]) to three-dimensional space.

Conclusions

The goal this paper has been to generalize the notion of Voronoi-diagrams to three-dimensions for open, connected, and bounded sets $\mathscr{F}$ that satisfy a special set of axioms, in the spirit of abstract data types. The generalization is done in such a way that the Findpath Problem for a sphere $\mathscr{M}$ moving in an admissible set $\mathscr{F}$ can be solved by a retraction method using the Voronoi-diagram of $\mathscr{F}$.

We tried to single out a set of properties of $\mathscr{F}$ that are indispensable for the proof of the Main Theorem. The properties of $\mathscr{F}$, i.e., the existence of a finite set contour($\mathscr{F}$) wih the required properties, may still be a little too restricting for some sets $\mathscr{F}$. For example, for circular cylinders the set contour($\mathscr{F}$) is not finite because each point on the axis of a circular cylinder is not uniquely projectable onto the cylinder. However, for the retraction of a path in $\mathscr{F}$ onto the Voronoi-diagram one may modify the path a little such that only finitely many of these points are on the path. Thus, one does not need the continuity of the function retr in these points. Elliptic cylinders (that are not circular) cause no troubles as bounding elements of $\mathscr{F}$. So it seems to be quite natural that also circular cylinders may be allowed in $\partial\mathscr{F}$. However, by our approach there is a wide class of sets $\mathscr{F}$ to which the retraction method is applicable.

We do not yet have an efficient algorithm to construct the Voronoi-diagram. However, a straight forward algorithm can be used for the construction; see [12]. It will be the work of future research to try to generalize efficient algorithms that construct the Voronoi-diagram in the plane to three dimensions.

APPENDIX

Proof of the Lemma on Faces and Curves. (a) We do the proof by a case analysis. Assume face$(A, B) \neq \varnothing$. (W.l.o.g. we do not consider the finitely many points that are not uniquely projectable onto some element in contour$(\mathscr{F})$.)

Case 1. $A \subseteq \bar{B}$. Then $F(A, B)$ is the surface containing A normal to B. Clearly, this is a face or a curve, depending whether A is a curve or a point, and whether B is a face or a curve.

Case 2. $B \subseteq \bar{A}$. Analogous to Case 1.

Case 3. $\neg$near(A, B).

Subcase a: A a face. It suffices to show that there is a continuous function from some open (in the topology of A) subset C of A whose image is $F(A, B)$.

For $q \in A$, let $l(q)$ be the straight line through q normal to A. We first show that $l(q) \cap F(A, B) \neq \varnothing$ for q in an open subset C of A. Because face$(A, B) \neq \varnothing$ (and, hence, $F(A, B) \neq \varnothing$) there is a $q' \in A$ such that $l(q') \cap F(A, B) \neq \varnothing$. Hence, for some open neighbourhood U' of q', $l(q') \cap F(A, B) \neq \varnothing$ for $q' \in U' \cap A$; hence, especially, there is some $p \in l(q')$: dist$(A, p) =$ dist(B, p).

For $p \in l(q)$ with dist$(A, p) =$ dist(B, p), let $n(p) \in \bar{B}$ with dist$(n(p), p) =$ dist$(B, p) =$ dist(A, p). Let $C = \{q \in A \mid pn(p)$ is normal to B, $n(p) \in B\}$. Clearly, C is open (in the topology of A). The function that maps each $q \in C$ onto $n(p)$, where $p \in l(q) \cap F(A, B)$ as constructed above (there is not necessarily only one such p) is continuous and its image is contained in B.

It remains to show that there are only finitely many elements in $l(q) \cap F(A, B)$ for each $q \in C$. We show that there is at most one element in $l(q) \cap F(A, B)$ on each side of A. Let $p_1, p_2 \in l(q) \cap F(A, B)$, dist$(A, p_1) <$ dist(A, p_2), $q_1 = n(p_1)$, $q_2 = n(p_2)$.

In case $q_1 = q_2$, dist$(p_2, q) =$ dist$(p_2, q_1) <$ (dist$(q, p_2) -$ dist$(q, p_1)) +$ dist$(q, p_1) =$ dist(q, p_2), a contradiction!

In case $q_1 \neq q_2$, dist$(q, p_2) =$ dist$(q_2, p_2) <$ (dist$(q, p_2) -$ dist$(q, p_1)) +$ dist$(q, p_1) =$ dist(q, p_2), a contradiction!

Subcase b: B a face. Analogous to the previous subcase.

Subcase c: A a curve, B not a face. Similar to Subcase *a*.

Subcase d: A, B both points. Then $F(A, B)$ is a plane, hence, a face.

(b) Let curve$(A, B, C) \neq \emptyset$. We do the proof again by a case analysis. Let A, B, C be pairwise distinct.

Case 1. near(A, B), near(A, C), near(B, C). Thus, w.l.o.g. $A \subseteq \bar{B}$, $B \subseteq \bar{C}$, $A \subseteq \bar{C}$; i.e., A is a point, B is a curve, and C is a face. curve(A, B, C) is the straight line through A normal to C. Clearly, this is a curve.

Case 2. near(A, B), near(A, C), $\neg$near(B, C).

Subcase a: $A \subseteq \bar{B}$, $A \subseteq \bar{C}$. Hence, $A \subseteq \bar{B} \cap \bar{C}$. For $p \in F(A, B)$: p is contained in a straight line through some $q \in A$ normal to B. The same holds for $p \in F(A, C)$. This entails that $F(B, C)$ contains at most finitely many elements, namely those that are not uniquely projectable onto B or onto C. Hence, curve(A, B, C) is finite.

Subcase b: $A \subseteq \bar{B}$, $C \subseteq \bar{A}$. Then $C \subseteq \bar{B}$, a contradiction to not near(B, C).

Subcase c: $B \subseteq \bar{A}$, $A \subseteq \bar{C}$. Then $B \subseteq \bar{C}$, a contradiction to not near(B, C).

Subcase d: $B \subseteq \bar{A}$, $C \subseteq \bar{A}$, $\bar{B} \cap \bar{C} = \emptyset$. Then curve$(A, B, C)$ is contained in the union of the intersection of two planes and (because almost all p are uniquely projectable onto A) at most finitely many other points.

Subcase e: $B \subseteq \bar{A}$, $C \subseteq \bar{A}$, $\bar{B} \cap \bar{C} \neq \emptyset$. Then $\bar{B} \cap \bar{C}$ is a point in $\bar{A}$. Hence, curve(A, B, C) is the straight line through $\bar{B} \cap \bar{C}$ normal to $\bar{A}$, and finitely many other points.

Case 3. near(A, B), $\neg$near(A, C), $\neg$near(B, C). W.l.o.g. assume $A \subseteq \bar{B}$. $F(A, B)$ is the surface containing A normal to B. We have shown in the proof of part (a) of this lemma that each ray in some $p \in A$ normal to B has at most one intersection point with $F(B, C)$. Because dist is continuous the set of all these intersection points is a subset of (the closure) of the union of the images of a continuous functions on $(0, 1)$.

Case 4. $\neg$near(A, B), $\neg$near(A, C), $\neg$near(B, C). $F(A, B)$ is the union of images of continuous functions on $(0, 1) \times (0, 1)$. Each ray normal to A in some $q \in A$ has at most one intersection point with $F(A, B)$ and at most one with $F(A, C)$. Because curve$(A, B, C) \neq \emptyset$ there is a $q \in A$ with $(l(q) \cap F(A, B)) \cap (l(q) \cap F(A, C)) \neq \emptyset$ (with the notation of part (a)). Let $p \in (l(q) \cap F(A, B)) \cap (l(q) \cap F(A, C))$. Assume $F(A, B) = F(A, C) = F(B, C) =$ curve(A, B, C) in some open neighbourhood V of p. (In V straight lines normal to A, B, C exist in each point. $F(A, B)$ partitions V into exactly two regions: one that contains all points in V that are closer to A than to B, and the other that contains all points in V that are closer to B than to A. Let $q_C \in C$ be such that pq_C is normal to C. Then $(\overline{pq_C} \cap V) - \{p\}$ is totally contained in one of these regions because $p \in F(A, B) = F(A, C) = F(B, C)$

(in V) and there must not be more than one intersection point of pq_C and $F(A, C)$, and of pq_C and $F(B, C)$. W.l.o.g. assume $\overline{pq_C} - \{p\}$ is contained in the region that is closer to B than to A. $\forall p' \in \overline{pq_C} - \{p\}$: $\text{dist}(p', C) < \text{dist}(p', B)$. Let $p' \in \overline{pq_C} - \{p\}$, $p' \in V$. Let $q_B \in B$ be such that $p'q_B$ is normal to B. W.l.o.g. let p' be such that $p'q_B \cap F(A, B) \in V$. $\overline{p'q_B} \cap V$ is totally contained in the region that is closer to B than to A because p' is in that region and the distance to B decreases when moving from p' to q_B. Thus, $\overline{p'q_B} \cup \overline{p'q_C}$ is totally contained in this region. $q_B \in B$, $q_C \in C$; i.e., $\text{dist}(q_B, B) = 0 > \text{dist}(q_B, C)$, $\text{dist}(q_C, C) = 0 > \text{dist}(q_c, B)$. Hence, there is some $p'' \in \overline{p'q_B} \cup \overline{p'q_C}$ with $\text{dist}(p'', B) = \text{dist}(p'', C)$ because dist is continuous. Because on each ray normal to C there is at most one point that has equal distance to B and C (as shown in part (a) of this lemma), one gets $p \in \overline{p'q_C}$, a contradiction. (Note that also in case C is a point each ray through C has at most one point with equal distance to B and C.

(c) Let p be on the bounding curve of some connected component C of vface(A, B), $A, B \in \text{contour}(\mathscr{F})$. W.l.o.g. we assume that p is uniquely projectable onto each element in contour$(\mathscr{F})$. Otherwise, take some point in a sufficiently small neighbourhood of p as new p. It suffices to show that the bounding curves of vfaces are contained in curves. The property on the distance to $\partial\mathscr{F}$ is satisfied because $p \in \text{vface}(A, B)$. We do the proof by a case analysis.

Case 1. near(A, B). W.l.o.g. let $A \subseteq \bar{B}$.

Subcase a: $p \notin F(A, B)$. Then p projects onto some q, $q \in \bar{A} - A$. Clearly, $\{q\} \in \text{contour}(\mathscr{F})$. Thus, $p \in \text{face}(\{q\}, A)$, and $p \in \text{face}(\{q\}, B)$. That is, $p \in \text{curve}(A', B', C')$ for some A', B', C'.

Subcase b: $p \in F(A, B)$. Then there is a $q \in A$ with pq normal to A. Furthermore, there is a $D \in \text{contour}(\mathscr{F})$ with $\text{dist}(p, D) = \text{dist}(p, A)$, and $\text{dist}(p', D) < \text{dist}(p', A)$ for $p' \in pq$, $p' \notin C$. Thus, there is also some $D \in \text{contour}(\mathscr{F})$ with this property, and for each such p' there is a $q' \in D$ with $p'q'$ normal to D. Clearly, $D \neq A$, $D \neq B$. Hence, $p \in \text{face}(A, D)$. Furthermore, there is some p'' in the neighbourhood of p such that there is a $q'' \in B$ with $p''q''$ normal to B, $\text{dist}(B, p'') = \text{dist}(q'', p'')$ and p'' is also projectable onto D. Hence, $\text{face}(B, D) \neq \emptyset$, and $p \in \text{face}(B, D)$. This entails $p \in \text{curve}(A, B, D)$.

Case 2. $\neg$near(A, B).

Subcase a: $p \notin F(A, B)$. Then p projects onto some q, $q \in \bar{A} - A$ (or $\bar{B} - B$). Clearly, there is some $D \in \text{contour}(\mathscr{F})$ with $q \in D$. $D \neq A$, $D \neq B$, otherwise near(A, B). $p \in \text{face}(A, D)$ by definition of face. Furthermore, $p \in \text{face}(B, D)$: If p projects onto some $q' \in B$ then $p \in \text{face}(B, D)$. If p projects onto some $q' \in \bar{B} - B$ then there is a p' in some neighbourhood of p that projects onto B and D, respectively. Thus, p', and also $p \in \text{face}(B, D)$. This entails $p \in \text{curve}(A, B, D)$.

Subcase b: $p \in F(A, B)$. Then there are $q \in A$, $q' \in B$ with pq normal to A, pq' normal to B. There is some $D \in \text{contour}(\mathscr{F})$ with $\text{dist}(p, D) = \text{dist}(p, A)$, and $\text{dist}(p', D) < \text{dist}(p', A)$ for some p' in some neighbourhood of p.

Subcase: near(A, D), near(B, D). Then $A \subseteq \bar{D}$. (Otherwise, dist$(r, D) \geqslant$ dist(r, A) for all r in some neighbourhood.) There is some p' in each neighbourhood of p that projects onto D. Hence, pq is normal to D; i.e., $p \in$ face(A, D). The same considerations can be done for B; i.e., $p \in$ face(B, D). This entails $p \in$ curve(A, B, D).

Subcase: $\neg$near(A, D), near (B, D). Then there is a p' in some neighbourhood of p with p' projects onto D and onto A. Hence, face$(A, D) \neq \emptyset$, and $p \in$ face(A, D). By near(B, D), $q' \in \bar{D}$, and pq normal to D (as in the previous subcase). Hence, $p \in$ face(B, D).

Subcase: near(A, D), $\neg$near(B, D). Analogous to the previous subcase.

Subcase: $\neg$near(A, D), $\neg$near(B, D). Then there is some $L' \in$ contour$(\mathscr{F})$ such that p projects onto D', $D' \subseteq \bar{D}$. By $\neg$near(A, D), one gets $\neg$near(A, D') and $A \neq D'$. Hence, $p \in$ face(A, D'). Let D'' be the same for B. Because p is uniquely projectable onto D, one gets $D' \cap D'' \neq \emptyset$; hence, $D' = D''$. By $p \in$ face(A, D'), $p \in$ face(B, D'), one gets $p \in$ curve(A, B, D').

(d) The proof that connected components of the vcurve's end only in vertices is done analogously. (In the case analysis one has additionally to consider C for $p \in$ vcurve(A, B, C).)

Proof of the Lemma on Continuity of retr. We show that retr-1 is continuous on $\mathscr{F}$ − 2-skel$(\mathscr{F})$ except at finitely many points and that retr-2 is continuous on $\mathscr{F}$ − vor$(\mathscr{F})$ except at the finitely many points $s_{A,B;C}$. (From these two facts one immediately gets the lemma.)

We first show that retr-1 is continuous except at points on the Voronoi-face and at points where V1 is not uniquely specified. (It is relatively easy to see that V1 is not uniquely specified only in finitely many points outside 2-skel$(\mathscr{F})$. The details are given in [12].) Let (p_n) be a sequence in $\mathscr{F}$ such that V1 is uniquely specified for each $p_n \notin$ 2-skel$(\mathscr{F})$ and (p_n) converges to some $p \in \mathscr{F}$ − 2-skel$(\mathscr{F})$. Let $q_n :=$ retr-1(p_n). One has to show that (q_n) converges to retr-1(p). For this it suffices to show that each converging subsequence of (q_n) converges to retr-1(p). So assume w.l.o.g. that (retr-1(p_n)) converges to some q.

Case. There are finitely many n with $p_n \in$ 2-skel$(\mathscr{F})$.

Then the convergence does not change if one takes the subsequence without those p_n. So assume w.l.o.g. that $p_n \notin$ 2-skel$(\mathscr{F})$ for all n. It is straight forward to see that the 2-skeleton is closed; see [12] for details. From this one gets $q \in$ 2-skel$(\mathscr{F})$. Let $A \in$ contour$(\mathscr{F})$ be such that V1$(p_n) \in A$ for infinitely many n; let (p_{n_i}) be such a subsequence with $p_{n_i} \in A$. It suffices to show that (q_{n_i}) converges to retr-1(p) (because (q_n) converges). By definition of V1 one gets (V1(p_{n_i})) converges to V1(p). V1$(p) \notin \bar{A} - A$ because $p \notin$ 2-skel$(\mathscr{F})$. (Otherwise, there is a $B \subseteq \bar{A}$, $V1(p) \in B$, $B \neq A$. Furthermore, $pV1(p)$ is normal to A. Thus, $p \in$ face(A, B).) Hence, the first intersection point of $pV1(p)$ and 2-skel$(\mathscr{F})$, say q', is the limes of retr-1(p_n). Let $B \in$ contour$(\mathscr{F})$ be such that $q_{n_i} \in$ vface(B) for infinitely many n_i.

Clearly, the first intersection point of $pV1(p)$ and 2-skel($\mathscr{F}$) is in vface(A, B), otherwise $V1(p) \notin A$. (Note that dist is continuous and A is the image of a continuously differentiable function.) However, there is only one intersection point on each ray through a point in A normal to A and face(A, B) (see the proof of the Lemma on Faces and Curves).

Case. There are infinitely many n with $p_n \in$ 2-skel($\mathscr{F}$).

By (p_n) converges to p and the fact that 2-skel($\mathscr{F}$) is closed, one gets that $p \in$ 2-skel($\mathscr{F}$); i.e., this case is not possible.

It remains to show that retr-2 is continuous on $\mathscr{F} - \text{vor}(\mathscr{F})$ except at points where V2 is not specified uniquely. (It is relatively easy to see that these are the finitely many points where $s_{A,B;C} = p$ for sme $A, B \in \text{contour}(\mathscr{F})$, $A \neq B$, C a connected component of vface(A, B).)

Let (p_n) be a sequence in 2-skel($\mathscr{F}$) such that V2 is uniquely specified for each $p_n \notin \text{vor}(\mathscr{F})$ and (p_n) converges to some $p \in \mathscr{F} - \text{vor}(\mathscr{F})$, $p \neq s_{A,B;C}$. Let $q_n :=$ retr-2(p_n). One has to show that (q_n) converges to retr-2(p). For this it suffices to show that each converging subsequence of (q_n) converges to retr-2(p). So assume w.l.o.g. that (retr-2(p_n)) converges to some q.

By a case analysis and similar arguments as in the first part of the proof, one shows that $q =$ retr-2(p). See [12] for details. ∎

Proof of the Lemma on Increasing Distances. dist($\partial\mathscr{F}$, p) $\leqslant$ dist($\partial\mathscr{F}$, retr-1(p)) is an immediate consequence of the definition of retr-1 (not depending on the concrete V1(p) chosen). (In the Lemma on Faces and curves we have shown that each $pV1(p)$ has an intersection with 2-skel($\mathscr{F}$). Furthermore, the distance to $V1(p)$ increases when moving on $pV1(p)$ from $V1(p)$ to p.) dist($\partial\mathscr{F}$, p) $\leqslant$ dist($\partial\mathscr{F}$, retr-2(p)) holds because, for $p \notin \text{vor}(\mathscr{F})$, retr-2($p$) is a point on V2($p$, A, B) on the same side of $s_{A,B;C}$ as p: If A, B are both points, or near(A, B) then $V2(p, A, B)$ is a straight line through p such that the distance to A increases when moving away from $s_{A,B;C}$ or p, respectively, i.e., when moving from p to retr-2(p). In case A, B are not both points and not near(A, B) the distance to $\partial\mathscr{F}$ increases, not necessarily strictly, when moving from p to retr-2(p) because retr-2(p) is on the other side of p as $s_{A,B;C}$ (or equal to p), $s_{A,B;C}$ is a point with minimal distance to A, and A and B have no maximal distance. For $p \in \text{vor}(\mathscr{F})$, retr-2($p$) $= p$. From this one gets dist($\partial\mathscr{F}$, p) $\leqslant$ dist($\partial\mathscr{F}$, retr(p)). Because retr-p is defined as a limes in p, for the p considered, if retr is not continuous in p, this relation also holds for retr-p. ∎

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